Introduction

Math can often seen in daily life, and piles of material which display a specific shape are frequently seen--the shape is a function of the angle of repose, which is the steepest angle a material will pile without avalanching. The *Angle of Repose unit plan* is a four lesson sequence which begins with a review of right angled triangles, which lays the groundwork for the investigation to come. The "artefact" is sand. A hands-on investigation studies the angle of repose for dry and damp sand, encouraging a macro-scale analysis of the phenomena using geometric reasoning. The investigation also reinforces data collection, the ability to calculate angles based on measurements, and presentation skills.

Unit Plan: Angle of Repose

Duration: 4 lessons, 1 standard class period each (50 minutes)

Prior Study: Pythagoras Theorem, use of calculator.

Curriculum Links: Pythagoras and Trigonometry

"Solve right-angled triangle problems including those involving ... angles of elevation and depression." (Year 10 Scope and Sequence, ACARA, 2012).

Objectives:

•To consolidate understanding of the trigonometric tangent function, and the concept that two variables (two side-lengths or a side-length and an angle) uniquely define the size and shape of a right triangle.

•To analyse the properties of a granular material's angle of repose with a hands-on investigation.

•To reinforce measurement skills and the practical use of tangent function.

•To promote the ability to "see" math in everyday objects.

•To promote positive interdependence in group work.

•To refine presentation skills with a data-centric report or poster.

Criteria[1]: Students can...

Identify the use of the tangent function to determine angles and/or length of the sides of a triangle, and an example of an angle of repose outside the context of a classroom.

[1] Verb criteria informed by the SOLO taxonomy (Biggs, 2003).

Describe the tangent function, and *Compare* mathematical reasons for differing angles of repose for granular materials.

Analyse and *Apply* mathematical methods for calculating angle, and explain causes of avalanching of granular materials.

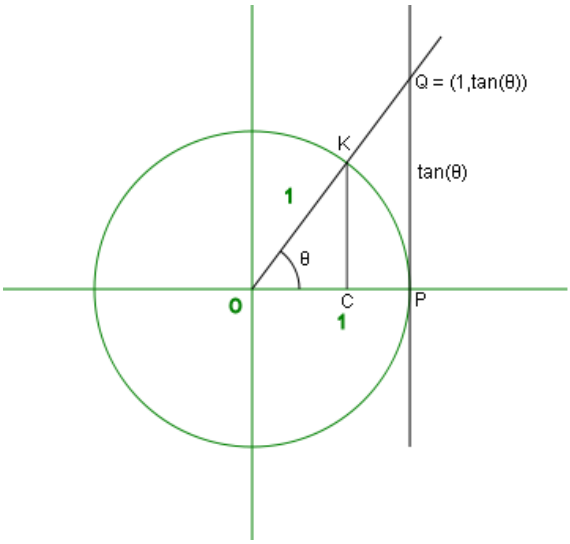
Formulate general strategies for angle measurements, and *Generalise* the concept of angle of repose of various materials by identifying relationship of material properties and other factors influencing the angle of repose.

Assessment: Informal assessment and scaffolding throughout, including investigation methodology. Formal assessment: tangent worksheet (Attachment 1), final project.

Lesson Sequence

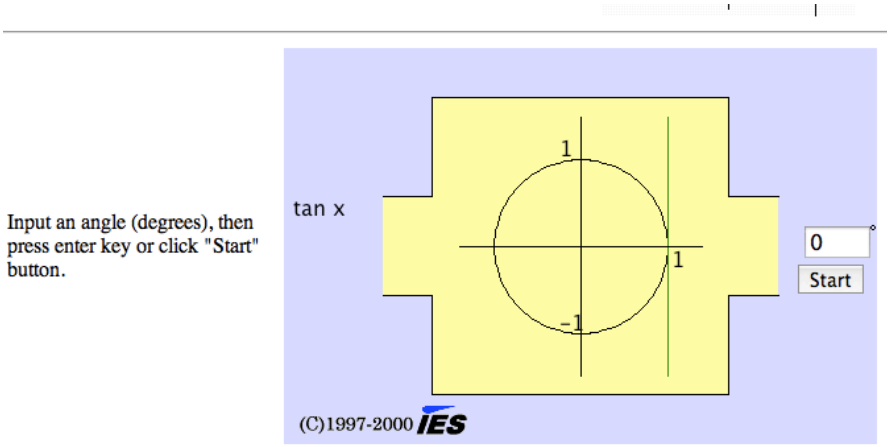
Lesson 1A (15 minutes, direct instruction)

Begin with review of Pythagorus Theorem ($a^2+b^2=c^2$) on whiteboard. Review the right triangle (draw one), and note that with the length of two sides known, we can find length of the third side. Emphasise the two "known" sides define the triangle's size *and* angles. Elaborate by asking, "what if, instead, we know one of the acute angles in a right angle triangle?" Draw a 30-60-90 triangle. Ask if the angle alone uniquely defines the triangle. Scaffold responses. Ask, "what else do we need to define the triangle?", and scaffold responses as the length of any one of the sides. Note that again, just as in the Pythagoras Theorem situation, we have two variables--an angle the length of one side, and with it we should be able to define the triangle. Ask how can we do this, and suggest ideas leading to the need for trigonometry. Discuss "Trigonometry" word origin: "triangle" and "measure". Draw a unit circle with an inscribed triangle (image right, Wikibooks, 2012). Provide the definition of tangent--the ratio of two sides (opposite and adjacent) of a right triangle. Note how vertical line is tangent to the circle in which the triangle is inscribed, linking to the trigonometric term. Draw a 45-45-90 triangle, and ask, based on the definition, what is the tangent of 45 degrees? Finally, scaffold awareness of how anything smaller than 45 degrees will be less



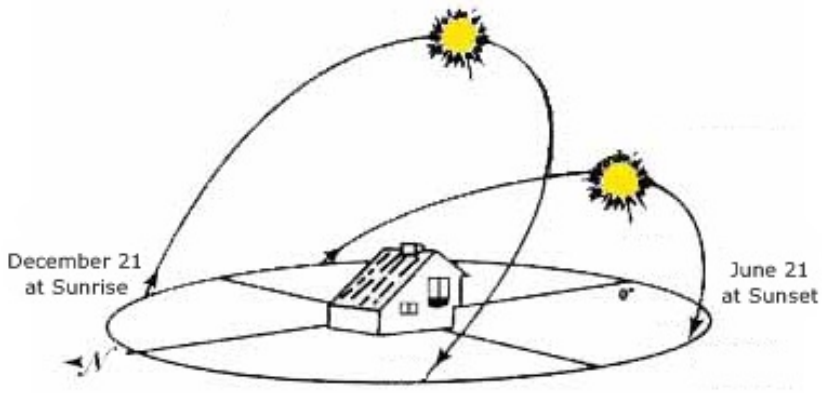
than one, anything more will be greater than 1. Provide a series of triangle examples, and without calculator, have students "guess" the tangent of angles by estimating the ratio of the two sides (Hancock, 2007). Finally, scaffold use of the arctan ($\tan^{-1}$) function on their calculators.

Extensions: ICT resource at: http://www.ies.co.jp/math/products/trig/applets/tanbox/tanbox.html (screenshot below).



Lesson 1B (25 minutes, hands-on investigation, sunny day required)

Materials: tape measure, math notebooks, computer access. Announce that we will measure the height of a nearby tree by measuring its shadow and by researching the sun's "altitude"--the angle of the sun to the horizon. Assign groups of three (see note on groupings below). Review path of the sun (image source: http://blogs.ozimut.org/?p=10).



Scaffold the use of solar altitude web page at http://www.ga.gov.au/geodesy/astro/smpos.jsp (Geoscience Australia, 2012) to find the altitude of sun for the current date and time. Set task: ensure that groups record the time of day when they measure the shadow length of a tree, and make a sketch of the investigation (tree, shadow, and sun) in their notebooks. Provide 10 minutes for students to measure shadow outside, then back to class for calculation using the tan function on their calculators.

Lesson 1C (review and worksheet, 10 minutes)

Record each group's height measurement of tree on whiteboard. Discuss reasons for variability in results. Reinforce tangent definition as ratio of the the opposite over adjacent sides of a right triangle. Announce next lesson's investigation on "Angle of Repose". Homework: Tangent Review Worksheet (Attachment 1).

Note on Groupings

For the purposes of the investigation, assign groups of three, with self-assigned roles:

•Engineer (ensures proper use of materials and collect sand from teacher).

•Recorder (organises tools for measurement and ensures data is collected properly).

•Mediator (ensure the group is working effectively and ensures whole team understands task--also is the only one to leave group to ask teacher and/or other groups for assistance).

The goal is to create group interdependence; Esmonde (2009) recommends three distinct roles--procedural, cognitive, and interpersonal--within each group. Yet emphasise that among each group, each member is accountable for performance of the team and can explain the process and results. Research by Chizik (1997) finds that optimal benefits derive from grouping medium-ability students together as members of homogenous groups, but grouping low- and high-ability students as members of heterogeneous ability groups, in order to promote high level verbal interaction and to "encourage children to make multiple hypothesis to answer their own questions and, then, to research which hypothesis is most plausible" (p. 61).

Lesson 2A (30 minutes)

Collect Tangent Review Worksheet (formally assessed).

Presentation on Angle of Repose (Attachment 2).

Introduce investigation task--measurement of angle of repose for dry and damp sand.

Lesson 2B (10 minutes)

In front of class, have two piles of sand--one dry, one damp--along with funnels and paper cups with holes and protractors. Gather students and allow hands-on initial experimentation with creating piles of sand. Provide a magnifying glass and note how a few particles can cause a chain reaction (avalanche), and how particles then travel down the sides to the bottom. Recall the ant lion video from the presentation--the purpose of the ant lion throwing sand upwards was not to attack the ant, but to initiate avalanches above

the ant. Note that the damp sand can be packed and formed with paper cups--scaffold methods on measuring the *rest* angle of repose (the angle of sand right after an avalanche); thus, scaffold techniques to drop clumps of damp sand to form a pile.

Lesson 2C (15 minutes)

Using same groups as in lesson 1B, at desks, students design investigation for measuring angle of repose for dry and damp sand. How will they create the piles? How will they measure too distinct methods of measurement? Announce assessment task will be a group poster or individual reports on the investigation. The report or poster needs to include at least the following five items:

1. sketch of each sand pile.
2. collected data presented clearly.
3. results of two measurements of each sand pile, and explanation of results, including comparison between dry and damp sand.
4. explanation and/or interesting fact about the angle of repose (see criteria).
5. an sketch or photo of an example of where else the angle of repose can be seen in "everyday" life. Provide criteria for assessment to students.

Homework: Outline steps of investigation and bring in any special equipment for experiment. Clarify that the experiment and the report/poster will need to be completed in one class period.

Lesson 3A (introduction, 3 minutes)

Query groups as to development of their investigation. Scaffold assignment of roles--speaker, manager, director. If groups are organised and have a plan, allow them to begin, one group at a time. If any groups have not developed a plan, scaffold with them to the point where they understand the task, and have a notion of each person's role.

Lesson 3B (investigation and group report, 32 minutes)

Resources: Pans or trays, protractors, spoons, paper cups, string, rulers, scissors. Provide about 1kg dry and 1kg damp sand to each group. Reinforce neatness. Scaffold observation of what happens when dry sand is added to pile--observe ensuing avalanche, and describe movement of sand (observation can be included in report). Ensure two methods of measurement for each pile (typically protractor and height/base dimensions). Clean up.

Lesson 3C (group work on report, 15 minutes)

Groups work on report or A3 poster.

Homework: complete presentation.

Lesson 4 (presentation of poster)

Each group presents their report to the class (5 minutes each).

Notes for teacher on Angle of Repose

The angle of repose for a granular material is a function of many factors, including particle size, shape, and density, and the coefficient of friction, and gravity. The cohesive effect of water has only relatively recently been quantified (Hornbacker, Albert, Albert, Barabasi & Shiffer, 1997). The interstitial liquid bridge effect (related to van der Waals forces) increases the angle of repose, but the "clumping" effect of moisture (long-range correlations among particles) does not appear to be fully understood--typically in dry granular substances, the correlations are short-range and solely between individual particles. The effects of saturation (where the angle of repose dramatically decreases resulting in, for example, landslides) is only empirically known. Because of the importance of angle of repose in civil engineering slope stability applications, the effect has been extensively studied[2], but no comprehensive theory on the angle of repose appears in the literature, making it an ideal investigation for inquiring minds.

The investigation can be extended with other mathematical concepts, such as finding the volume and surface area of the conical shape of a sand pile. Overall, the goal is to help students "see" math in everyday sightings, with the specific aim of encouraging links to mathematical thinking, especially when playing in the sand on beach days.

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[2] Van Burkalow (1945) outlines five experimental results on angles of repose.
--Angle of repose varies inversely with size of fragments in perfectly sorted materials, but directly in those imperfectly sorted.
--Angle of repose varies inversely with density of fragments.
--Angle of repose varies directly with the angularity, roughness, and degree of compaction.
--Angle of repose varies inversely with height of fall of material on free cones
--Angle of repose varies directly with increase of moisture up to the saturation point but inversely beyond that.

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