Philosophical Statement

The fundamentals of numeracy, as language is to literacy, are innate (Rips, Bloomfield, Asmuth, 2008). From the earliest age, babies begin to make sense of the world though the recognition of number and space, with continuing conceptualisation in the pre-school years. Paralleling the development of oral language, mathematical literacy progresses with concepts related to quantification and spatial awareness, and is both represented and shaped by mathematical language (e.g. "more than", "closer", etc.). The link between language and numeracy becomes more tangible with milestones such as the cognitive comprehension of one-to-one correspondence (stable number order cardinality), and continues with the parallel development of mathematical concepts and mathematical language as the child progresses[1].

Another way to consider the mathematical concept/language link is in terms of informal/formal knowledge (Russell & Ginsberg, 1984). Innate mathematical knowledge develops informally, while the curriculum and assessment relies primarily on the communication of formal mathematical knowledge. Teaching maths--numerical, spatial, graphical, statistical and algebraic disciplines (AAMT, 1997)--recognises and builds on learner's innate knowledge in order to assist the learner to construct and formalise their mathematical knowledge. In addition to a "profound, flexible, and adaptive knowledge" (Ma, 1999 as cited in Van de Walle, Karp & Bay-Williams, 2010, p. 9), the teacher requires a perceptive and empathetic awareness of the learner's mathematical developmental phase in both concept and language in order to begin a dialogue with the student. Pedagogy begins with an analysis of prior knowledge and respect for student language[2], then developmentally continues with problem-solving experiences within the learner's Zone of Proximal Development (ZPD) (Vygotsky, 1978). Whenever possible, therefore, the teacher initiates lessons with physical objects (to build innate knowledge)[3], then with conceptual scaffolding and modelling of formal language, guiding numeracy assimilation and accommodation of increasingly abstract conceptualisation. The degree of conceptual understanding informs the appropriate timing for the introduction of abstract algorithms, as dependence on algorithms can result in learners losing "some of their

[1] One of the pedagogical challenges, of course, is the duality of specific language expressions, i.e., the Bible's "Be fruitful and multiply," promotes a conception that multiplication always leads to an increase in quantity.
[2] At this diagnostic level, with Socratic questioning, the teacher investigates student logic used to justify and explain understanding of a particular concept in a non-evaluative manner.
[3] Recognition of the progression of abstraction is useful, e.g. start counting exercises with tangible contextual items, such as apples, then with counters, then with representations of counters, etc. This assists with developing a "grounded appreciation of context" (AAMT, 1997, p. 15).

capacity for flexible and creative thought" (Board and Davenport, 1993 as cited in Clarke, 2005, p. 95).

One of the challenges for teachers, perhaps without explicit memory of their own steps and stages of learning, is the ability to deconstruct mathematical knowledge into its components while retaining the connecting linkages to the original broader concepts. This process begins with a review of "big ideas" appropriate for the learner (Van de Wall, Karp, Bay-Williams, 2010); for example, one of the "big ideas" for operations is the relationship of addition and subtraction. By focusing on a specific concepts and linkages, the teacher can target the learner's development of efficient mathematical thinking and strategies that lead to procedural fluency and adaptive reasoning. Consistently reinforcing links to the learner's existing cognitive schema builds confidence in math by allowing learners "to realising that he [or she] has been thinking mathematics all along (Bruner, 1995, p. 333).

Finally, a collaborative classroom model enhances learning. The classroom environment can be "competitive, individualistic, or collaborative" (Loreman, Deppeler & Harvey 2005, p. 155). The collaborative model, with small heterogeneous groups of five students, assists "pupils [to] become numerate though purposeful interpersonal activity based on interactions with others" (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997, p. 18), and encourages the expression of multiple strategies which can lead to stronger understandings. Planning for multiple entry points and differentiating lesson plans (Van de Walle & Lovin, 2006) provides opportunities for both reinforcement and extension.

Teaching Resources

The *Overview of Diagnostic Map: Number* (Appendix 1) (First Steps, 2012) provides verbal descriptions of mathematical thinking phases in the early years, beginning with the "emergent phase" and progressing to "operating phase", representing increasing levels of abstraction. Such resources are essential diagnostic tools to identify student understanding, as well as providing guidance for appropriate ZPD experiences. There are many resources in the diagnostic category (including assessment tools); a comprehensive list has been compiled by Australian Council for Educational Research [ACER] (Forster, 2009). Knowledge of the Piagetian successive stages of development leading to the formal operational stage also fits in this realm of resources. For example, learners in the pre-operational stage (typically to year 2), have "incomplete" logic (Piaget, 1972/2008) in terms of reversibility, conservation, and transitivity; therefore the design of lessons should reflect these aspects and primarily focus on activities such as problem-solving tasks with physical materials to build quantification and object-characterisation skills. Similarity, in the concrete operational stage (typically to year 5/6), hands-on ("concrete") activities and contextual word-problems (rather than a pure algebraic equation, for example) would be appropriate as learners have yet to reach the formal operational stage of abstract reasoning (Ojose, 2008).

The second resource for appropriate pedagogy is represented by a handout presented at the University of Tasmania (Chick, 2012) entitled "Operations". The handout is a taxonomy of addition and subtraction "situations", classified into four categories: "change-add to (join)", "change--take from (separate)", "combine (part-part-whole)", and "compare". Though there are other ways to classify addition and subtraction problems, having a specific taxonomy as a reference can provide the stimulus for creating a variety of word problems to expand the learner's heuristic abilities. A similar taxonomy can be created for identifying mental computation strategies (i.e. "split tens", "sequential", "complementary addition")(McIntosh, 2003) which can be used a resource in identifying and scaffolding informal logic. Appropriate language is also a key pedagogical attribute and aligned with this type of resource, which helps deconstruct "big ideas" for a consistent pedagogy. For example, the use of "base-ten" language (i.e. "four tens and seven" for 47) when teaching place value concepts can provide explicit numeracy links (Van de Walle, Karp, Bay-Williams, 2008).

The third resource that relates to my philosophy of teaching is represented by the Multi-Base Arithmetic Blocks (MAB) (Appendix 3). Experiential exploration of physical manipulatives (and well designed virtual ones) builds innate mathematical knowledge by providing a means to creatively formulate and discover mathematical investigations, often with multiple solution paths. Language is also developed: when using MAB blocks, the concept of "trading", when applied to later formal algorithms, offers a conceptual link. In the pre-operational stage, playful activity with a variety of materials (i.e square cubes, nesting cups, shape puzzles, sectioned objects that fit together) can be encouraged with prompts and questions (Eisenhauer and Feikes, 2009). In the concrete-operational stages of development, more complex manipulatives combined with explicit initial scaffolding provides the informal groundwork for subsequent formal reasoning. In contrast, "experiential deficits in informal knowledge are the most robust predictors of difficulties acquiring formal math skills (Griffin, Case, & Siegler, 1994 as cited in Methe, 2009, p. 10).

Misconceptions in Mathematical Reasoning

In contrast to mistakes, a misconception in mathematical reasoning can occur when a learner, in constructing mathematical knowledge, extends a mathematical concept or "rule" in an fallacious manner. With modern constructivist teaching pedagogy, which encourages the learner to build on innate knowledge, the obligation for the teacher to be vigilant in identifying misconceptions becomes especially critical. The primary pedagogy involves a diagnosis focused on providing students opportunities to comprehensively explain their logic. Once a misconception is identified, the teacher then provides learning opportunities and experiences that encourages self-recognition of the misconception, which then prompts the student to modify their cognitive schema by accommodating the revised conception. Built on the Piagetian concept of disequilibrium, in order for a learner to abandon a misconception and accommodate conceptual exchange, there needs to be dissatisfaction with the misconception (Hewson, 1992). Note that the misconception diagnostically informs the level of student understanding; as Osborne expressed, "We must start where the child is" (1982).

The misconception I have chosen to address relates to the "law of small numbers" (Tversky & Kahneman, 1971, p. 105) which is the misconception that a small sample represents a larger group. In contrast, the "law of large numbers" reflects the accurate inference that the larger the sample, the larger the probability of the sample representing the whole. The misconception can arise from an intuitive extension of the quantitative part-whole concept of numbers, and is interrelated to other misconceptions; Fischbein and Schnarch (1997) outline seven common probabilistic misconceptions:

1. Representativeness (i.e., the overestimation of an event's likelihood due to its similarity to the parent form).
2. Negative and positive recency (i.e., coin tosses have "memory").
3. Compound events (i.e., similar outcomes from a two-coin toss are not differentiated).
4. Conjunction fallacy (i.e., personal bias influences perception of likelihood)
5. Sample size (i.e., variation from mean is misjudged in smaller sample).
6. Availability (i.e., the likelihood of more 'available' events, such as a isolated event portrayed in the media, is overestimated).
7. Falk Phenomenon (i.e., the effect of a gap in time on probability).

The "law of small numbers" is related to the misconception of representativeness (Fischbein & Schnarch, 1997) and also linked to the misconception of recency, as the notion that a random generated set can have "memory" is similar to the expectation that consistent patterns will be seen in small samples.

I chose this difficult misconception because it highlights the importance of reinforcing mathematical language and informal mathematical reasoning as the groundwork for statistical numeracy, as the formal analysis of probability requires abstract combinatorial logic[4], and as Confrey (1990) argues, preferentially initially presented with a subjective approach to reinforce intuition. Probabilistic misconceptions can detrimentally affect critical thinking skills when an individual is presented with data and sampled information. The misconception can be diagnosed[5] with the "hospital problem":

> In a small town at a small hospital, 10 babies are born per day on average. In the nearby city at a large hospital, 50 babies are born per day on average. Each hospital has a celebration when 60% of the births on a given day are girls; they call this a "girl day". Which hospital--the small one or the large one--is more likely to have a girl day, or would they both be equally likely to have a girl day? (adapted from Fischbein & Schnarch, 1997)[6].

Most children have the misconception that both hospitals have equal likelihood for girl days, with the larger hospital being the second choice (Fischbein & Schnarch, 1997)[7].

Diagnostic assessment continues with an understanding of the level of development. Beginning with Piaget and Inhelder in 1951 (Davies, 1965), researchers have linked probabilistic reasoning with cognitive development stages, yet no complete theory has emerged (Way, 2003). Way distinguishes three stages:

Stage 1: *Non-probabilistic thinking.* Students in this stage have minimal understanding of randomness, are reliant on visual comparison, and are unable to order likelihood.

Stage 2: *Emergent Probabilistic Thinking.* Students in this stage can recognise impossible or equally likely sample spaces, and use additive proportional reasoning to make generalisations about likelihood.

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[4] Because of its dependence on abstract reasoning, the formal analysis of probability would be more appropriate when learners have reached the formal operational stage.

[5] This problem would only be appropriate for the upper-primary grades, but a similar problem could be created that was more contextual for younger students. For example: *Ann has a box of chocolates, filled with jelly-filled chocolates and caramel-filled chocolates, which all look the same, but there are less caramel-filled chocolates than jelly-filled ones. Bob picks two chocolates and hopes that both are caramel, while Cara picks six chocolates and hopes that all six will be caramel. Do you think Bob or Cara will be more likely to have their hopes realised, or are they both equally likely? Explain why.*

[6] The original "hospital problem" was posed by Kahneman and Tversky in 1972.

[7] The smaller hospital, of course, is the correct answer, due to the smaller sample size.

Stage 3: *Quantification of Probability.* Students in this stage begin to recognise the relationship between randomness and likelihood, use multiplicative proportional reasoning, and are developing a language of probability (e.g. correct use of "chance" and "more likely") (Way, 2003).

Addressing the misconception is dependent on the diagnostic assessment and involves exposure to activities which reinforce statistical numeracy intuition[8], scaffolded with the development of probabilistic language. A three tier approach is recommended:

1. Basic understanding of statistical terminology.
2. An understanding of statistical language and concepts when they are embedded in the context of a wider social discussion.
3. A questioning attitude one can assume when applying more sophisticated concepts to contradict claims made without proper statistical foundation (Watson and Moritz, 2000, p. 45).

In particular, Watson and Moritz focus on developing the technical understanding of the language of sampling though a series of explorations that differentiate the common usage of "sampling" (in reference to a homogeneous whole such as a "sample of cheese" or a "sample of blood") with the statistical usage through explorations of group variability, such as the height or weight of students, which highlight the relative representativeness of larger samples.

Probability lesson plans are informed by the "big ideas" already discussed, and "simulation is a technique used for answering real-world questions ... in which an element of chance is involved" (Van de Walle, Karp and Bay-Williams, 2010, p. 456). POE (predict-observe-explain) investigations with ICT manipulatives (see NLVM, 2012) are appropriate. Using contextual examples drawn from student's interests assists with engagement. Parallel lessons in equiprobability would also be of benefit in addressing sampling misconceptions. In conclusion, appropriate probability pedagogy at each stage of a child's development is important for the progressive construction of statistical numeracy, providing an increased intuition of situations where statistical bias can occur, as well as "motivate the questioning attitudes required of future citizens" (Watson & Moritz, 2000).

[8] Intuition in this context is the secondary intuition developed by systematic instruction that are "acquisitions that have all the characteristics of [primary] intuitions but . . . are formed by scientific education, mainly in school" (Fischbein, 1975 as cited in Greer, 2001, p. 18).

REFERENCES

Australian Association of Mathematics Teachers [AAMT]. (1997). *Numeracy=everyone's business.* The Report of the Numeracy Education Strategy Development Conference, May 1997. Adelaide: AAMT. Retrieved from: http://www.aamt.edu.au/Professional-reading/Numeracy

Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997). *Effective teachers of numeracy in primary schools*. In T. Muir (Ed.), Primary and early childhood mathematics pedagogy (pp. 18-19). Frenchs Forest, NSW: Pearson Education Australia.

Bower, B. (2010). Gambling on experience. *Science News 177*(9), pp. 26-29.

Bruner, J.S. (1995). On learning mathematics. *The Mathematics Teacher 88*(4), pp. 330-335.

Chick, H. (2012). *Operations (Workshop 7).* Handout provided to students at the University of Tasmania, April 18, 2012.

Clarke, D.M. (2005). *Written algorithms in the primary years: Undoing the 'good' work?* From Making mathematics vital : proceedings of the twentieth biennial conference of the Australian Association of Mathematics Teachers, Adelaide.

Confrey, J. (1990). Review of the research on student conceptions in mathematics, science, and programming. *Review of Research in Education 16*, pp. 3-56.

Davies, C.M. (1965). Development of the probability concept in children. *Child Development 36* (3), pp. 779-788.

Eisenhauer, M.J. & Feikes, D. (2009). Dolls, blocks and puzzles: Playing with mathematical understanding. *Young Children 64* (3), pp. 18-24.

First Steps (2012). *Overview of diagnostic map: Number.* Retrieved from: http://www.pearsonprofessionallearning.ca/firststepsmath/pdf/number_simplified_map.pdf (First Steps: http://www.stepspd.com/au/resources/firststepsmath.htm)

Fischbein, E., & Schnarch, D. (1997). The Evolution with Age of Probabilistic, Intuitively Based Misconceptions. *Journal for Research in Mathematics Education 28*(1), pp. 96-105.

Forster, M. (2009). *Literacy and numeracy diagnostic tools: An overview.* Retrieved from the Australian Government Department of Education, Employment and Workplace Relations [DEEWR] website: www.deewr.gov.au/Schooling/Programs/LiteracyandNumeracy/Documents/DiagToolsReport.pdf

Greer, B. (2001). Understanding probabilistic thinking: The legacy of Efraim Fischbein. *Education Studies in Mathematics 45*, pp. 15-33.

Loreman,T., Deppeler, J & Harvey, D. (2005). *Inclusive Education: A practical guide to supporting diversity in the classroom.* NSW: Allen & Unwin.

Hewson, P. (1992). *Conceptual change in science teaching and teacher education.* Paper presented at the National Center for Educational Research, Documentation, and Assessment, Madrid, Spain.

McIntosh, A. (2003). *Developing computation.* Australian Government Department of Education, Science and Training. Retrieved from: www.dest.gov.au/NR/.../tas_developing_computation_booklet.pdf

Methe, S. (2009). "Big ideas": Missing pieces in early mathematics assessment. *National Association of School Psychologists 38* (3), pp. 1, 10-11.

National Library of Virtual Manipulatives [NLVM]. (2012). *Data Analysis & Probability.* Retrieved from http://nlvm.usu.edu/en/nav/topic_t_5.html

Osborne, R. (1982). Science Education: Where do we start? *The Australian Science Teachers Journal, 28*(1).

Piaget, J. (2008). Intellectual evolution from adolescence to adulthood. *Human Development 51*. pp. 40–47. (Reprint of original work published in 1972 in *Human Development 15*).

Rips, L.J., Bloomfield, A. & Asmuth, J. (2008). From numerical concepts to concepts of number. *Behavioural and Brain Sciences 31(6)*, pp. 623-642.

Tversky, A., & Kahneman, D. (1971). Belief in the law of small numbers. *Psychological Bulletin 76* (2), pp. 105-110.

Van de Walle, J.A., & Lovin (2006). *Teaching student centred mathematics: Grades 3-5.* In T. Muir (Ed.), Primary and early childhood mathematics pedagogy (pp. 116-122). Frenchs Forest, NSW: Pearson Education Australia.

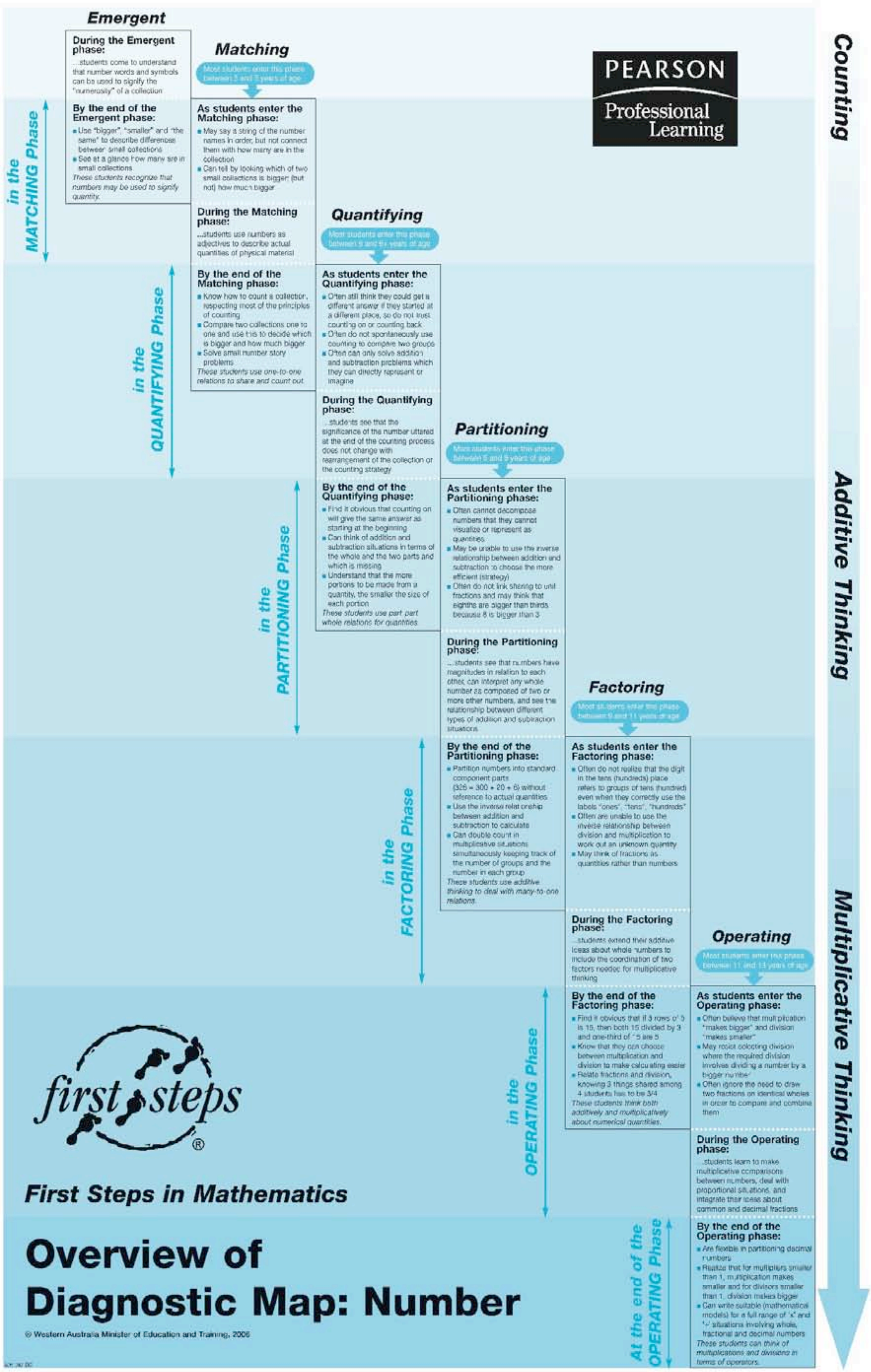
Van de Walle, J.A., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and middle school mathematics: Teaching developmentally (7th ed.).* Boston: Allyn & Bacon.

Vygotsky, L.S. (1978). *Mind in Society.* Retrieved from: http://books.google.com.au

Watson, J..M. & Moritz, J.B. (2000). Developing concepts of sampling. *Journal for Research in Mathematics Education 31*(1), pp. 44-70.

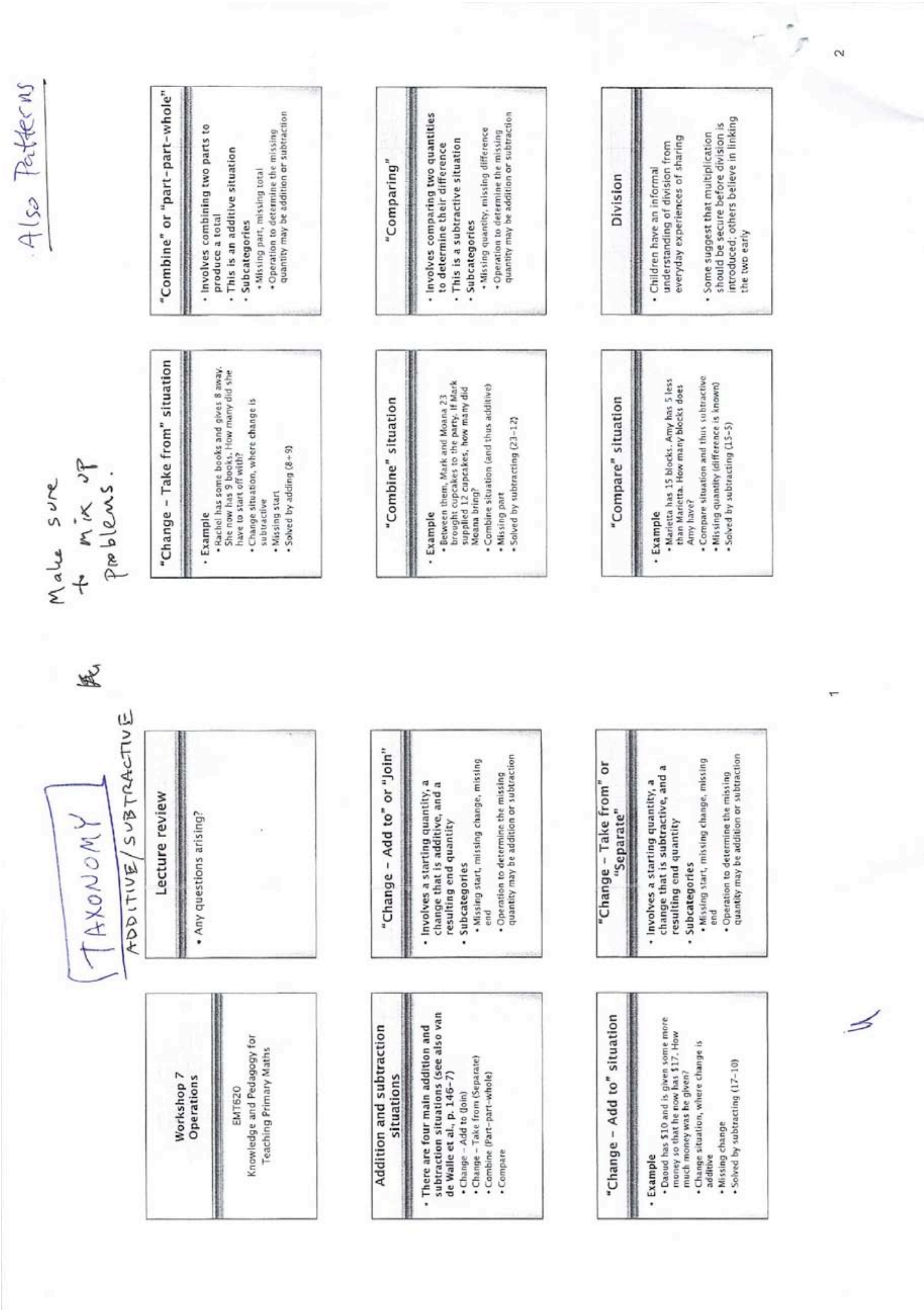
Way, J. (2003). *The development of young children's notions of probability.* From Proceedings of Cerme: European Research in Mathematics Education.

Appendix 1



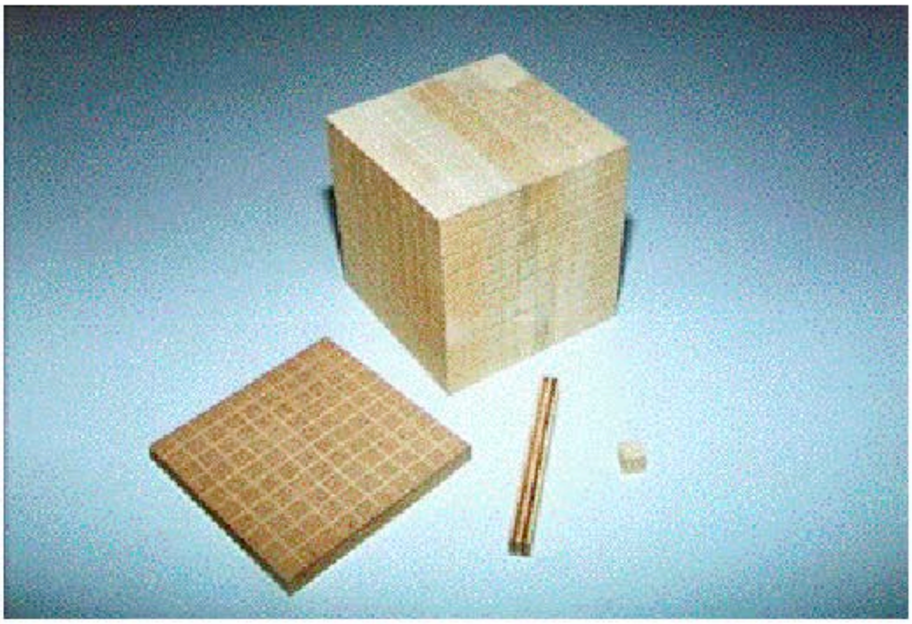
Source: www.deewr.gov.au/Schooling/Programs/LiteracyandNumeracy/Documents/DiagToolsReport.pdf

Appendix 2



Source: Chick, 2012.

Appendix 3



Overview of use of MAB

Base 10 MAB are found in many primary schools and are commonly used in teaching the *representation of* and *operations with* **whole numbers**. To use them to represent decimals, students need to be convinced that blocks do not always have the same value. For example, the large cube may represent 1, rather than 1000. (See below for how this might be done). MAB can then be used in teaching the *representation of* and *operations with* **decimal numbers**. Activities such as making and naming numbers, ordering by size, estimation, addition, subtraction, multiplication and division (as repeated addition and subtraction) in the new decimal number realm can be given a concrete embodiment. This assists in avoiding just giving lists of rules for dealing with decimals.

Some students may think it is babyish to use MAB for decimals because they have used them in younger grades. In this case, consider using **Linear Arithmetic Blocks** (LAB) or Area Cards instead. They look different but can do the same things.

Instructions on using MAB correctly in teaching whole number operations (addition, subtraction, multiplication and division) can be found in standard primary mathematics education textbooks (Booker et al, 1997; English and Halford, 1995).

Concrete materials such as MAB are of great assistance to demonstrate WHY addition and subtraction algorithms work and the meaning of notation. However, very careful attention must be given to making the link between the concrete and the symbolic. Teachers need to be very careful that children see the parallels between moving the blocks and carrying out operations on numbers.

It is appropriate to discuss (before or after starting to introduce decimals) that the blocks can be grouped together to make new and larger units. For example, 10 large cubes can be put together to make a *superlong*, 10 of these would make a *superflat* and 10 of these would make a *supercube*, and that the process *need never stop!*

Building up these base ten links between columns is a crucial conceptual underpinning for whole numbers and for decimals.

Source: University of Melbourne Graduate School of Education Science and Mathematics Education
http://extranet.edfac.unimelb.edu.au/DSME/decimals/SLIMversion/teaching/models/mab.shtml