Rationale

The world can be examined mathematically, and this unit is designed to expose the principles of system dynamics, stability, instability, and chaos that students see in everyday life though a series of lessons and games. The objective is to encourage students to consider and see natural processes mathematically, and the assessment will be based on the student's ability to identify, explain, analyse, and create mathematical models.

The first two lessons are introductions; lesson one introduces the difference equation concept and its notation. Lesson two introduces chaos to demonstrate that even a relatively simple equation with only one variable can be highly dependent on initial conditions and exhibit a high degree of complexity. The unit then progresses to graphical representations and mathematical modelling of systems. Two lessons are then provided to learn and use the NetLogo (Northwestern University, 2012) programming environment, followed by a lesson on economic modelling of supply and demand. Finally, groups of students choose and develop a mathematical model of their choosing for the final assessment based on criteria and a rubric.

One of the primary objectives of this unit is to identify and distinguish between two notions of time: continuous and discrete. Continuous time is the reality, but discrete mathematics provides the tools for analysing real-world models (Scheinerman, 2000). Distinguishing between the two can involve cognitive dissonance; for example, though the "rule" for calculating Fibonacci numbers is quite simple, Leonardo da Pisa's original notion of rabbit growth can be challenging due to the theoretical delay of each rabbit's initial reproduction rate. In the Fibonacci sequence, both notions of time are present: the growth (continuous) that takes place between the intervals, and the measurement (discrete) at each interval. Cognitive structuring of these two notions of time is a goal of this unit.

The unit requires a strong foundation in algebraic linear equations and the representation of equations as graphs. Basic concepts, such as percentages, will be further scaffolded within lessons. Differential equations do not appear, but the concepts of calculus are inherent in the difference equations[1]. The unit covers a number of overlapping Year 10 Australian

[1] e.g.: the differential $dx/dt = ax(t)$ is equivalently expressed as a difference equation:
--> $(x_{t+1}-x_t)/h= ax_t$ (h is the step size)
--> $x_{t+1}= x_t + (ha)x_t$ (ha becomes the 'growth rate' r over a period of time)
--> $x_{t+1}= x_t + rx_t$ (in this case, the simple Malthus growth equation).

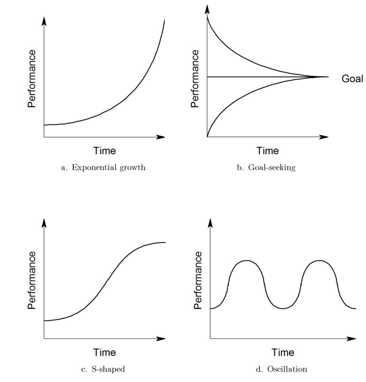
Curriculum (ACARA, 2012) concepts, including linear and non-linear relationships (ACMNA235), data representation and interpretation (ACMSP252), the compound interest concept (ACMNA229), and solving equations using algebraic, graphical, and digital method (ACMNA237).

Many physical, biological, chemical, economic dynamic processes are observed and measured in discrete increments, and an intuitive conceptual understanding of how each successive increment is dependent on the preceding prior instances can be beneficial to subsequent understanding of the differential equations that describe continuous behaviour of systems (Fisher, 2011). Philosophically, one could argue that the natural world is an evolutionary expression of the proximate preceding phenomena, rather than from the interval from the origin, as is expressed by representations of phi (e.g. patterns seen in sunflowers, nautilus shells, pine cones, and galaxies). Certainly, the digital world functions in discrete intervals, and this unit uses the computer to model and visualise feedback loops, oscillations, equilibrium, boundaries, exponential growth and decay by modelling a variety of real-world situations. By allowing students to hypothesise relationships and create interactive computer simulations, they can test the veracity of their models and reinforce mathematical intuition and perceptions (Spector, 2000).

UNIT PLAN Lesson 1. Introduction to difference equations.	
Objective and Activities	**Resources, Assessment**
Objective: introduce difference equation concepts and mathematical notation. Intro puzzle--students provide next number in series: 5,10,15,20,25,.... 1,4,9, 16, 25, ... 1,2,6,24,120, 720,... •Introduce Fibonacci (Leonardo da Pisa) numbers. Emphasise special cases x_0, x_1, then simple rule. Focus on NOTATION and generalise: $x_n=x_{n-1}+x_{n-2}$. •Contrast to prior series problems (based on position in series) and Fibonacci--not as simple to create "formula" for x_n based solely on the value of n. •For $x_n=2x_{n-1} + 1$ with initial conditions 1, 0, -1, -2, calculate first four terms. Extrapolate trajectory and emphasise concept of BOUNDED vs. unbounded (+/- infinity). •Have students find the first four terms for: $x_n = -1/2\ x_{n-1} - 3/2$ (result will converge to -1 in an alternating manner). $x_n = (x_{n-1})^2$ (result will converge or diverge depending on $\lvert x_0 \rvert < 1$) Provide time, then discuss bounded/unbounded trajectories for each equation based on the initial values for the difference equations. Tipping point concept. •Student challenge: Self-replicating robots (Fibonacci sequence) (see Appendix lesson 1).	Whiteboard. Images of nautilus shell, sunflower, galaxy. Formative assessment of robot challenge problem.
Take home problem, scaffolded in class: What is the difference between 12% annual interest compounded monthly (1% per month) vs. 12% compounded annually? Students to graph the two different interest rate methods.	Formative Assessment: Student can calculate subsequent values in a difference equation series.
Bonus homework--next item in this series (Conway): 1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7. . . Answer: a(n) = a(a(n-1)) + a(n-a(n-1)) [a(1)=a(2)=1]	

Lesson 2: instability and stability based on initial conditions.	
Objective and Activities	**Resources, Assessment**
Objectives: •sensitivity to initial conditions. •intuitive meaning of equations. The class begins with a physical demonstration of a double pendulum. Students will attempt to repeat a motion by starting the double pendulum in a specified position, recording number of loops in a 20 second period. Next, an introduction of history of chaos theory and how it differs from Newtonian determinism. Students investigate the logistic equation: $x_n=Rx_{n-1}(1-x_{n-1})$ The equation will be intuitively analysed by discussing the positive feedback of the first term, and the negative feedback of the second. Next, groups provided with different values of the growth rate R (same x_0 =0.1) and calculate the first 20 values. R<1 --> 0 R>1 and R<3 --> single value. R=3 and R = ~ 3.56 --> periodic values. R> 3.57, non-periodic values. ICT demo of variability of initial value. Discuss implications of results--why does the growth rate affect the pattern of outcomes? Introduce use of Excel to investigate the above difference equation. •Fractal presentation (briefly introduce/review imaginary numbers)--[Mandelbrot $x_n=(x_{n-1})^2 + c$]. Emphasis on the trajectory and bounds based on value of c, and that the Mandelbrot fractal is actually a graph. Review and reiterate difference equations and interest rate homework.	Double Pendulum (http://chaoticpendulums.com) (or build one in shop). ICT version: http://www.myphysicslab.com/dbl_pendulum.html See also: http://www.andrewclem.com/Chaos.html#Logistic.html Initial value demo (R=3.65) at http://johnmiddendorf.com/chaosDemo/chaos.html Excel spreadsheet template with difference equations supplied to students. Mandelbrot fractals (interactive and static WITH labeled axes; e.g. http://commons.wikimedia.org/wiki/File:Mandelbrot_Leminiscates_1_coords.png).

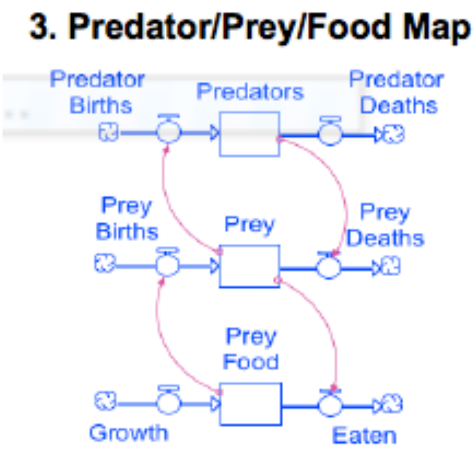
Lesson 3: Graphing review

Objective and Activities	Resources, Assessment
Objective: Thinking in terms of difference equations and introduction of relationships between independent variables. Begin with iPad seismometer--iPad on table, everyone jumps at once--> visual representation of vibration. Discuss and analyse simple model of a spring in terms of difference equations, e.g. $f_n = -f_{n-1} + a$ Next, review of graphs (four graphs). Scaffold understanding of the relationship of discrete points to the prior discrete points along the curves/lines. Ask what each of these graphs might be representing--provide time for groups to create “story”. Then, brainstorm with class to create possible difference equations for the graphs. Introduce new time-based notation: e.g. $P_t = P_{t-1} + rP_{t-1}$	iPad on smart board. Whiteboard.  Figure 1.2 Characteristic patterns of system behavior (Source: Kirkwood, 1998)
Fun time: Show few pages of the Lorax story on iPad, then, provide children’s books to groups, e.g. The Lorax and Uno’s Garden. Have groups model two or three variables in the books, e.g. Uno’s: graph plants and buildings vs. time. Emphasise shape and position of graph elements. Next, discuss relationships between the different variables. How can we express these relationships graphically (see appendix) and mathematically? Brainstorm and plug in simple values of time to test veracity of brainstormed relationships. Keep all brainstormed equations on board; of all possibilities brainstormed, gain consensus on “best” mathematical model, e.g. Truffula trees = f(Thneeds) (allow stochastic rules and special cases for values of t). Student challenge: In groups, identify and describe three to five major variables of a chosen dynamic system. Scaffold ideas (e.g. carbon cycle/global warming, student popularity, traffic, food production, etc.). Students identify and express interrelationships (+/-) among named variables. Students sketch feedback loops on whiteboard.	Lorax book on iPad--show a few pages on smart board to identify variables--ask: what can we graph? Formative assessment on relationships of variables, and clarity of graph, positioning and trends of curves and lines, as well as basic graph elements: title, axes (see Appendix lesson 3). Formative assessment of identifying relationships of variables and use of difference equations.

<table>
<tr><th colspan="2">Lesson 4: Introduction to System modelling.</th></tr>
<tr><th>Objective and Activities</th><th>Resources, Assessment</th></tr>
<tr><td>Objective: introduce system modelling concepts used in many simulation software and games.

Begin with review of interest rate problem, focus on intuitive expression of difference equations. Show that it can also be expressed as $M_n = M\ (1+ \text{interest rate})^{n}$.</td><td></td></tr>
<tr><td>Next, ask students what games they play, noting simulation games. Ask, “how do they work?” Then, groups play natural selection game on computer http://phet.colorado.edu/en/simulation/natural-selection. Predator-prey worksheet (Appendix lesson 4).

Brainstorm with class as to HOW the predator-prey simulation works.
Create a visual representation of relationships, e.g.:



(Source: Creative Learning Exchange, 2012).

Scaffold process to write mathematical equations, (emphasising that the computer is programmed and processing a mathematical relationships--what do the equations look like?).</td><td>Computer with Java enabled for each group of 3 students.

Alternate predator prey version here: http://forio.com/simulate/cle/l4predatorpreylevc/simulation/ (uses carrying capacity).

Whiteboard.</td></tr>
<tr><td>Introduce Lotka-Volterra Difference equations (used in many predator prey simulation models). See Appendix lesson 4b.

Groupwork-- using provided Excel templates, investigate various variables of Lotka-Volterra difference equations (Appendix lesson 4b).</td><td>Excel Spreadsheet Template (attached).

Formative assessment of variable variation and intuitive explanation of results.</td></tr>
</table>

Lesson 5-6: NetLogo programming.	
Objective and Activities	**Resources, Assessment**
Objective: NetLogo (Northwestern University, 2012) programming. Two lessons will be provided to learn NetLogo which can be installed on a Mac or a PC (Java required); the lessons will begin with examining the provided models, then move into the sample code. Ideally, students will learn the NetLogo language code (Appendix Lesson 5-6) and program their own model; however, it is also possible to investigate variations of the existing models using a command line interface. Both programming and manipulating variables will be scaffolded in these lessons.	NetLogo user manual at http://ccl.northwestern.edu/netlogo/docs/NetLogo%20User%20Manual.pdf

Lesson 7: Modelling the supply demand price difference equation.	
Objective and Activities	**Resources, Assessment**
Begin with whole class supply and demand game (Holt, 1996). Introduce supply and demand curves (Appendix Lesson 7) and scaffold intuitive interpretation. Create supply and demand curves for class game. Review linear equation slope and x-y intercept concepts. Introduce "cobweb pricing model", common in farming commodity models, which affects the supply of goods due to the theoretical expectation of the prior year's price (Appendix Lesson 7). Scaffold derivation of price equation per Fulford, Forrester & Jones (1997): $P_t=-aP_{t-1} + b$ (Reminder notes: a is the slope of the supply curve divided by the slope of the demand curve, b is the x-intercept of the demand curve minus the x-intercept of the supply curve divided by the slope of the demand curve).	Also see variants on economic game: http://www.socialstudiesforkids.com/articles/economics/supplyanddemand1.htm and cyrilmorong.com/Game.doc
Have students conjecture the possible result of the pricing difference equation for various values of a and b (scaffold intuitive meaning of these terms by relating them to the supply demand curves), followed by the students modelling equation in Excel or NetLogo. A report on the modelling of the pricing DE will be due and graded prior to the final project in order to provide individual feedback on the expectations of final project.	Graded assessment on individual report of modelled price difference equation (criteria and rubric will be similar to the final project).

Final Project--mathematical simulation assignment.			
Simulation of real-world dynamic system Students are to work in small groups to identify, describe, analyse, and generate a computer model of a real-world dynamic system. Examples include population models, carbon cycle, global warming, popularity of music bands, friendship networks, biological systems, etc. The model should incorporate feedback loops and/or multiple variables. Time will be allotted in several subsequent classes to scaffold assignment and explore other models available on the web and in literature. The assignment will be due in two weeks from the final class of the unit (lesson 7). **Criteria for assignment:** The student can: 1. Identify variables and relationships of a real-world process and apply mathematical reasoning. 2. Create and analyse a computer simulation using modelled mathematical relationships of the real-world process. **Rubric:**			
Criteria	*Exceptional*	*Proficient*	*Pass*
1	As per proficient, AND student justifies and identifies major assumptions made in the model. The nature of each relationship and all feedback loops are clearly identified in the diagram and description.	As peer pass, AND student graphically illustrates the nature of the variable relationships using diagrams and other media which provide the basis for the computer model.	Student describes a real-world dynamic system and outlines key mathematically related variables, and identifies relationships and dependencies of each variable.
2	As per proficient, AND student formulates insightful analysis of the validity of the model and if it could be further investigated to enhance a decision making process.	As per pass, AND student uses NetLogo or similar rich environment to simulate mathematical model and display results, reflects and justifies the modelled relationships, and evaluates the results.	Student uses Excel to model and graph simulation. At least three scenarios are compared and contrasted in the report, with justifications and explanations of variations of initial values and equations. All algorithms are clearly presented.

REFERENCES

Australian Curriculum Assessment and Reporting Authority [ACARA]. (2012). *The Australian curriculum v. 3.0: Mathematics foundation to year 10 curriculum.* Retrieved from: http://www.australiancurriculum.edu.au

Creative Learning Exchange (2012). *System dynamics & systems thinking in K-12 education.* Retrieved from: http://clexchange.org

Fulford, G., Forrester, P., Jones, A. (1997). *Modelling with Differential and Difference Equations.* Cambridge: Cambridge University Press.

Fisher, D.M. (2011). "Everybody thinking differently": K–12 is a leverage point. *System Dynamics Review (27)* 4 pp. 394-411.

Florio, D. (2012). Introduction to chaos. *Polygeek.* Retrieved from: http://polygeek.com/3116_flex_introduction-to-chaos-through-the-bifurcation-graph

Holt, C. (1996). Classroom games: trading in a pit market. *Journal of Economic Perspectives, 10* (1), pp.193-203.

Kirkwood, C. (1998). System dynamics methods: A quick introduction. *Arizona State University.* Retrieved from: http://www.public.asu.edu/~kirkwood/sysdyn/SDIntro/SDIntro.htm

Northwestern University. (2012). *NetLogo.* Center for Connected Learning and Computer-Based Modeling. Evanston, IL. Retrived from : http://ccl.northwestern.edu/netlogo/.

Scheinerman, E.R. (2000). Continuous vs. discrete mathematics. *The American Mathematical Monthly(107)* 6, p. 540.

Shahin, M. (2010). Explorations in elementary mathematical modeling. *Acta Didactica Napocensia (3)* 2, pp. 89-100.

Spector, J. M. (2000). System dynamics and interactive learning environments: Lessons learned and implications for the future. *Simulation & Gaming (31)* 4, pp. 528-535.

University of Houston. (2012). *Difference Equations.* Retrieved from: www.bauer.uh.edu/rsusmel/phd/MR-13.ppt

Zager, L. (2012). Fabulous Fractals and Difference Equations. *MIT Blossoms.* Retrieved from: http://blossoms.mit.edu/videos/lessons/fabulous_fractals_and_difference_equations

Appendix Lesson 1

Robot Problem

At noon, you create a self-replicating robot, which takes an hour to learn to self-replicate itself, then one hour to create a new self-replicating robot; thereafter, it creates a new robot every hour. Each new robot also takes an initial hour to learn to self-replicate, then creates a new robot every hour. How many robots are there at 5pm? 6pm? 7pm?

Try to create a graphic version of the growth of the robot population with a set of simple rules to apply at the end of each hour.

<u>Example rules to scaffold:</u>

Draw an outline of a box for the first robot.

At the end of each hour:

•if the box is empty, then half-fill the box.

•if the box is filled in, then add a empty box as a branch.

•if the box is half-filled, then fill the box.

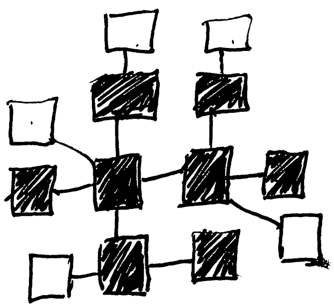
Tally the count.

Repeat.

- if empty, fill ½
- if full, add □
- if ½, then fill
- COUNT

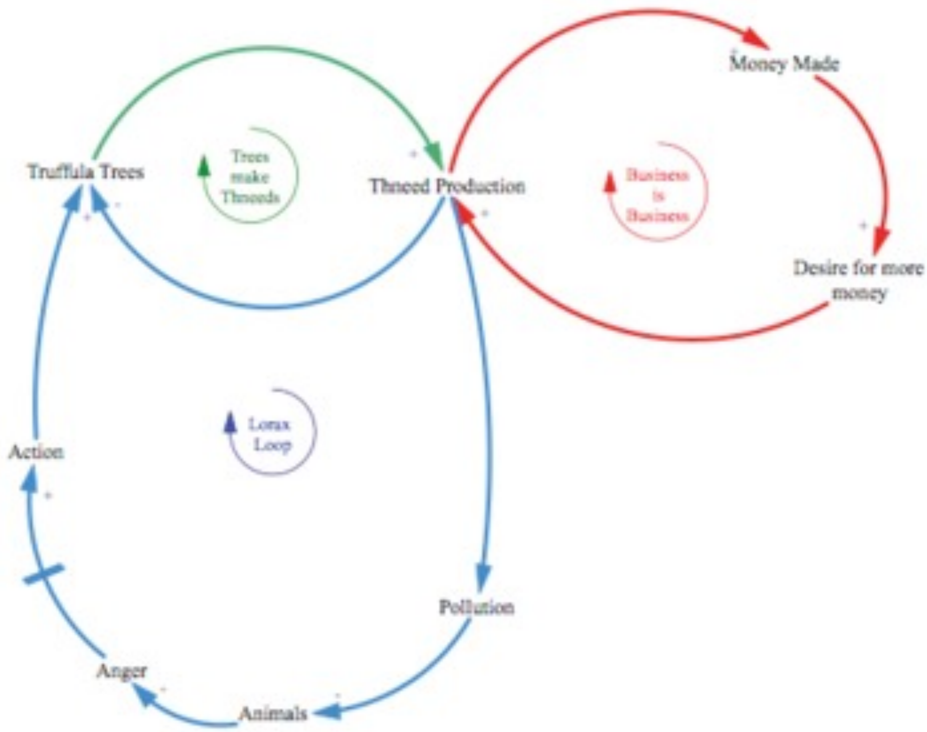
TALLY

0	1	2	3	4	5	6	7
1	1	2	3	5	8	13	



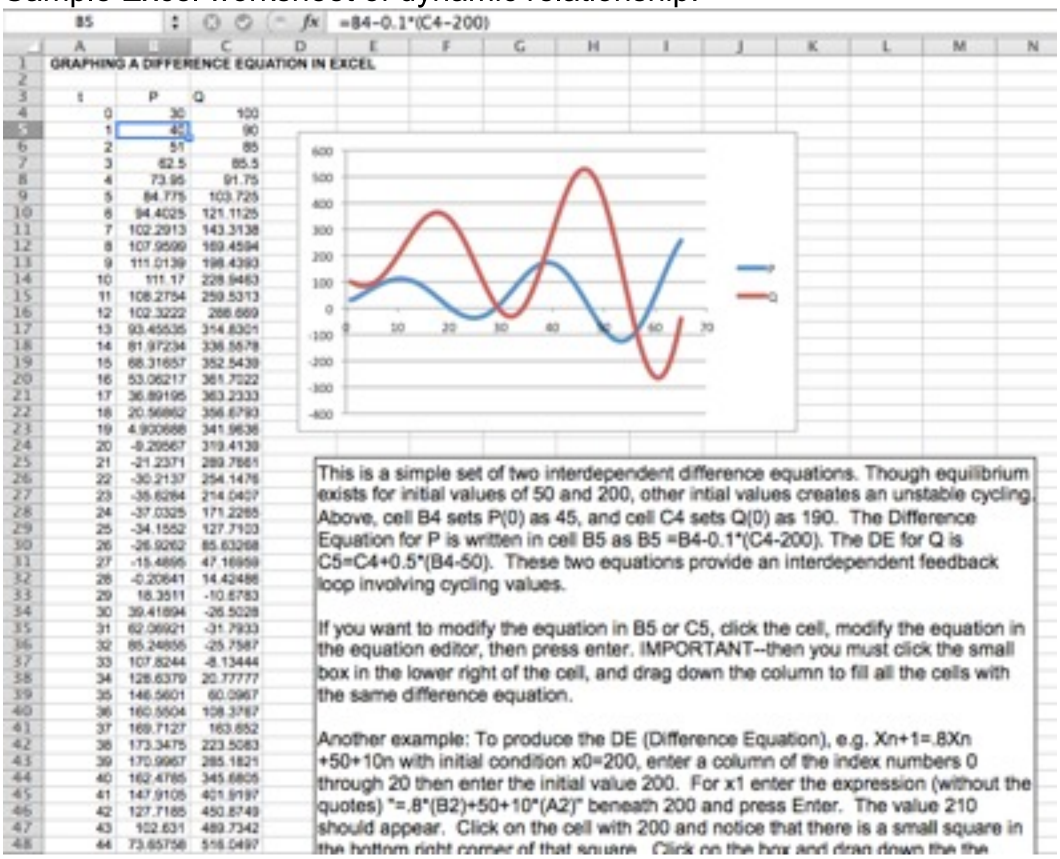
Document the recognition of patterns. e.g --> $x_n = 2(x_{n-2}) + x_{n-3}$

Appendix Lesson 3. Sample feedback loop diagram for the Lorax.



(Source: Creative Learning Exchange, 2012)

Sample Excel worksheet of dynamic relationship:



B5 =B4-0.1*(C4-200)

GRAPHING A DIFFERENCE EQUATION IN EXCEL

t	P	Q
0	30	100
1	45	90
2	51	85
3	62.5	85.5
4	73.95	91.75
5	84.775	103.725
6	94.4025	121.1125
7	102.2913	143.3138
8	107.9599	169.4594
9	111.0139	198.4393
10	111.17	228.9463
11	108.2754	259.5313
12	102.3222	288.669
13	93.45535	314.8301
14	81.97234	336.5578
15	68.31657	352.5439
16	53.06217	361.7022
17	36.89195	363.2333
18	20.56862	356.6793
19	4.900688	341.9636
20	-9.29567	319.4139
21	-21.2371	289.7661
22	-30.2137	254.1476
23	-35.6264	214.0407
24	-37.0325	171.2265
25	-34.1552	127.7103
26	-26.9262	85.63268
27	-15.4895	47.16959
28	-0.20641	14.42486
29	18.3511	-10.6783
30	39.41694	-26.5028
31	62.06921	-31.7933
32	85.24856	-25.7587
33	107.8244	-8.13444
34	128.6379	20.77777
35	146.5601	60.0967
36	160.5504	108.3767
37	169.7127	163.652
38	173.3475	223.5083
39	170.9967	285.1821
40	162.4765	345.6805
41	147.9105	401.9197
42	127.7165	450.8749
43	102.631	489.7342
44	73.65758	516.0497

This is a simple set of two interdependent difference equations. Though equilibrium exists for initial values of 50 and 200, other intial values creates an unstable cycling. Above, cell B4 sets P(0) as 45, and cell C4 sets Q(0) as 190. The Difference Equation for P is written in cell B5 as B5 =B4-0.1*(C4-200). The DE for Q is C5=C4+0.5*(B4-50). These two equations provide an interdependent feedback loop involving cycling values.

If you want to modify the equation in B5 or C5, click the cell, modify the equation in the equation editor, then press enter. IMPORTANT--then you must click the small box in the lower right of the cell, and drag down the column to fill all the cells with the same difference equation.

Another example: To produce the DE (Difference Equation), e.g. Xn+1=.8Xn +50+10n with initial condition x0=200, enter a column of the index numbers 0 through 20 then enter the initial value 200. For x1 enter the expression (without the quotes) "=.8*(B2)+50+10*(A2)" beneath 200 and press Enter. The value 210 should appear. Click on the cell with 200 and notice that there is a small square in the bottom right corner of that square. Click on the box and drag down the the

Appendix Lesson 4 --Phet Simulation--warmup to understanding simulations.
(source: http://phet.colorado.edu/en/simulation/natural-selection)

Natural Selection Lab- PhET Simulation

Designing The Experiment
In this Lab you will be controlling the mutations and environment of a population of rabbits. Your will create two hypotheses and design an experiment to test each one. Your hypothesis will follow the format where you fill in the (...) with your own ideas and reasons.

I hypothesize that (select a rabbit phenotype) rabbits will be (more/ less) likely to survive under (type of selective factor) within the (select type of environment) environment, because..... (explain how their trait will help them to survive or not)

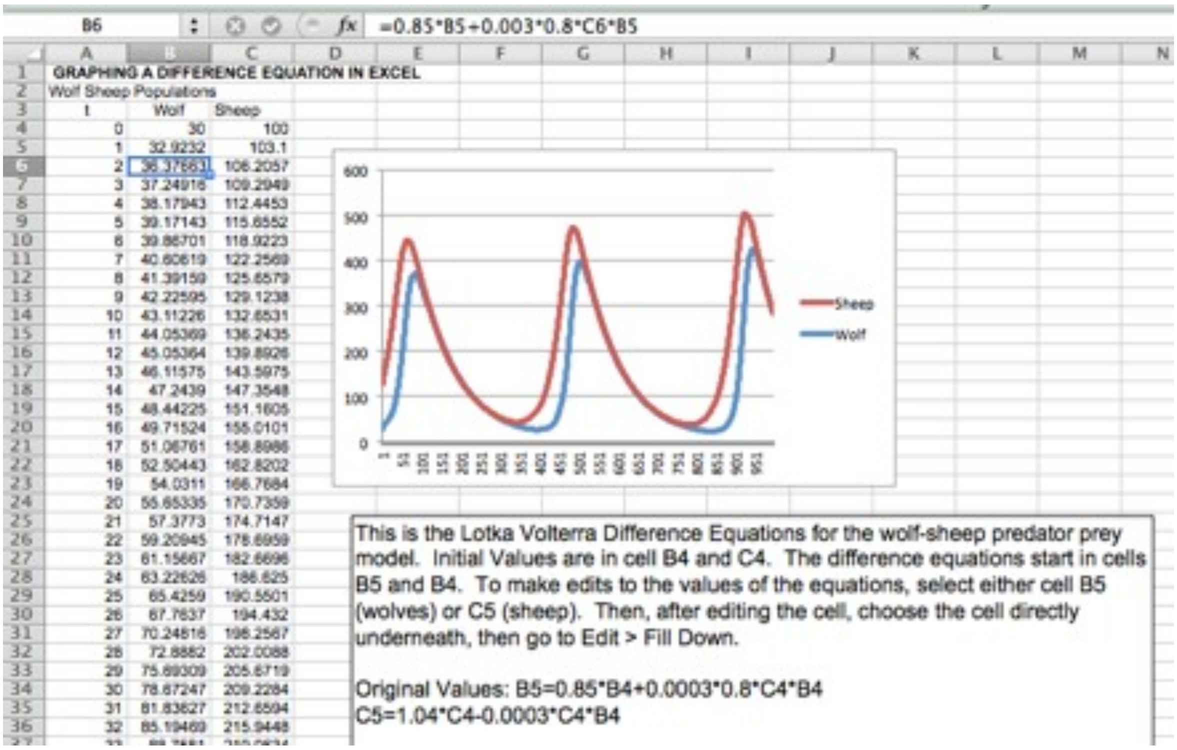
For each experiment you must have a control (no mutation) and fill in the following chart:

Experiment and Hypothesis	Pheno type	Selective Factor	CONTROL Group Initial Population at F3	CONTROL Group Final Population	Experment Group Initial Population at F3	Experiment Group Final Population	Conclusion/ Observation

- For each of the experiments, begin by adding a friend and a mutation. Wait until the F3 generation before adding the selective factor. After adding the selective factor let the simulation run for another 3 or 4 generations.
- Use the population numbers from the chart to get you numbers for the table, remember you can zoom in and out on the chart to get more accurate reads.
- Repeat for experiment 2.

Post-Lab Questions
1. Based upon your evidence from the simulation what conclusion are you able to make about each of the three different types of phenotypes in rabbits?
2. What happens to animals that cannot compete as well with other animals in the wild?
3. Sometimes animals that are introduced into an area that they never lived in before, out-compete and endanger resident species, why do you think this happens?
4. If only one species is considered the "fittest", why do we still have so many variations among species. Why do some birds have very long pointy beaks, while other birds have short flat beaks?
5. How do you think diseases can affect natural selection?
6. How does this simulation mimic natural selection? In what ways does this simulation fail to represent the process of natural selection.

Appendix Lesson 4b: Lotka Volterra Difference Equations.



B6 =0.85*B5+0.003*0.8*C6*B5

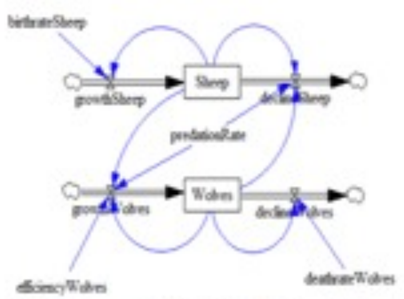
	A	B	C
1	GRAPHING A DIFFERENCE EQUATION IN EXCEL		
2	Wolf Sheep Populations		
3	t	Wolf	Sheep
4	0	30	100
5	1	32.9232	103.1
6	2	36.37663	106.2057
7	3	37.24916	109.2949
8	4	38.17943	112.4453
9	5	39.17143	115.6552
10	6	39.86701	118.9223
11	7	40.60619	122.2569
12	8	41.39159	125.6579
13	9	42.22595	129.1238
14	10	43.11226	132.6531
15	11	44.05369	136.2435
16	12	45.05364	139.8926
17	13	46.11575	143.5975
18	14	47.2439	147.3548
19	15	48.44225	151.1605
20	16	49.71524	155.0101
21	17	51.06761	158.8986
22	18	52.50443	162.8202
23	19	54.0311	166.7684
24	20	55.65335	170.7359
25	21	57.3773	174.7147
26	22	59.20945	178.6959
27	23	61.15667	182.6696
28	24	63.22626	186.625
29	25	65.4259	190.5501
30	26	67.7637	194.432
31	27	70.24816	198.2567
32	28	72.8882	202.0088
33	29	75.69309	205.6719
34	30	78.67247	209.2284
35	31	81.83627	212.6594
36	32	85.19469	215.9448

Above: image of Excel spreadsheet that will be supplied as template to students.

Lotka-Volterra Difference Equations: Sheep & Wolves

1. Original Model: $P(t) = P(t-1) + rP(t-1)$
2. $P_s(t) = P_s(t-1) + r_s P_s(t-1) - kP_s(t-1)P_w(t-1)$
3. $P_w(t) = P_w(t-1) + r_w P_w(t-1) + ekP_s(t-1)P_w(t-1)$
4. r_s: birth rate of sheep with no predators (>0).
5. r_w: death rate of wolves with no prey. (<0).
6. $P_s(t-1)P_w(t-1)$, encounters between wolves and sheep.
7. k: the rate at which wolves kill sheep they encounter.
8. e: efficiency of turning eaten sheep into a new wolf.

(Source: University of Houston, 2012)




```
Sheep=  INTEG  (growthSheep  -  declineSheep,
      initial value = 100)
Wolves= INTEG (growthWolves - declineWolves,
      initial value = 30)
birthrateSheep = 0.04
predationRate = 0.0003
efficiencyWolves = 0.8
deathrateWolves = 0.15
declineSheep = predationRate * Sheep * Wolves
declineWolves = deathrateWolves * Wolves
growthSheep = birthrateSheep * Sheep
growthWolves  =  efficiencyWolves  *  Wolves  *
      Sheep * predationRate
FINAL TIME  = 350 Units: years
TIME STEP  = 0.0078125
```

System Dynamics model of Predator-Prey-interaction. Left: Vensim model, right: used parameters.

Above: Code used in Predator Prey models (source: http://www.vensim.com)

Appendix Lesson 5-6: Sample NetLogo code

```
;; System dynamics model globals
globals [
  ;; constants
  sheep-birth-rate
  predation-rate
  predator-efficiency
  wolf-death-rate
  ;; stock values
  sheep
  wolves
  ;; size of each step, see SYSTEM-DYNAMICS-GO
  dt
]
;; Initializes the system dynamics model.
;; Call this in your model's SETUP procedure.
to system-dynamics-setup
  reset-ticks
  set dt 0.0010
  ;; initialize constant values
  set sheep-birth-rate .04
  set predation-rate 3.0E-4
  set predator-efficiency .8
  set wolf-death-rate 0.15
  ;; initialize stock values
  set sheep 100
  set wolves 30
end
;; Step through the system dynamics model by performing next iteration of Euler's method.
;; Call this in your model's GO procedure.
to system-dynamics-go
  ;; compute variable and flow values once per step
  let local-sheep-births sheep-births
  let local-sheep-deaths sheep-deaths
  let local-wolf-births wolf-births
  let local-wolf-deaths wolf-deaths
  ;; update stock values
  ;; use temporary variables so order of computation doesn't affect result.
  let new-sheep max( list 0 ( sheep + local-sheep-births - local-sheep-deaths ) )
  let new-wolves max( list 0 ( wolves + local-wolf-births - local-wolf-deaths ) )
  set sheep new-sheep
  set wolves new-wolves
  tick-advance dt
end

;; Report value of flow
to-report sheep-births
  report ( sheep-birth-rate * sheep
  ) * dt
end

;; Report value of flow
to-report sheep-deaths
  report ( sheep * predation-rate * wolves
  ) * dt
end

;; Report value of flow
to-report wolf-births
  report ( wolves * predator-efficiency * predation-rate * sheep
  ) * dt
end

;; Report value of flow
to-report wolf-deaths
  report ( wolves * wolf-death-rate
  ) * dt
end

;; Plot the current state of the system dynamics model's stocks
;; Call this procedure in your plot's update commands.
to system-dynamics-do-plot
  if plot-pen-exists? "sheep" [
    set-current-plot-pen "sheep"
    plotxy ticks sheep
  ]
  if plot-pen-exists? "wolves" [
    set-current-plot-pen "wolves"
    plotxy ticks wolves
  ]
end
```

Appendix Lesson 7: Economic Supply Demand Cobweb Model.

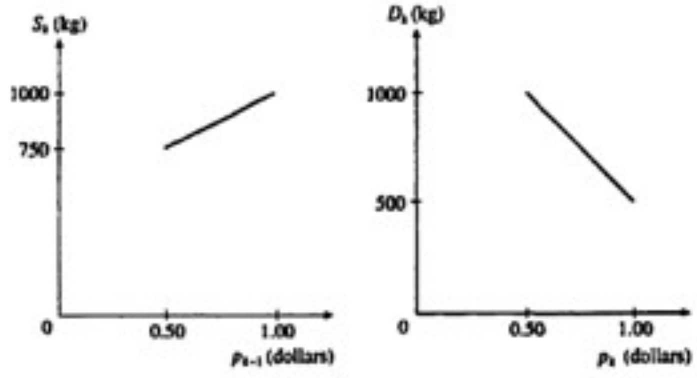


Fig. 8.3.1. Hypothetical supply and demand graphs for potatoes.

Solution. *The assumption (a) is satisfied because the supply S_k is a linear function of the price p_{k-1} and, from the graph, S_k increases when p_{k-1} increases. Similarly, assumption (b) is satisfied because D_k is a linear function of p_k and D_k decreases if p_k increases.*

It is easy to write down explicit formulae for the linear functions implicit in the graphs. In each case we read off the slope from the graph ('rise over run') and then add a constant to the RHS of the formula to make the graph pass through one of the chart points. This gives

$$S_k = 500p_{k-1} + 500,$$
$$D_k = -1000p_k + 1500.$$

The assumption (c) translates to $D_k = S_k$ and hence from the last two equations

$$p_k = -\frac{1}{2}p_{k-1} + 1 \qquad (k = 1, 2, 3, \ldots).$$

This is the required difference equation for the price.

(Derivation of price difference equation. Source: Fulford, Forrester,Jones 1997).

$$P_n = aP_{n-1} + b$$

where $a = -m_s / m_d$ and $b = (c_d - c_s)/m_d$.

Activity 3.1: Consider the supply and demand model (3.4) in the following cases:

Case 1: $m_s = 0.25$, $m_d = 0.5$, $c_s = 2$, $c_d = 8$, and $P_0 = 10$

Case 2: $m_s = 0.5$, $m_d = 0.5$, $c_s = 1$, $c_d = 8$, and $P_0 = 8$

Case 3: $m_s = 0.6$, $m_d = 0.5$, $c_s = 1$, $c_d = 7.6$, and $P_0 = 6.5$

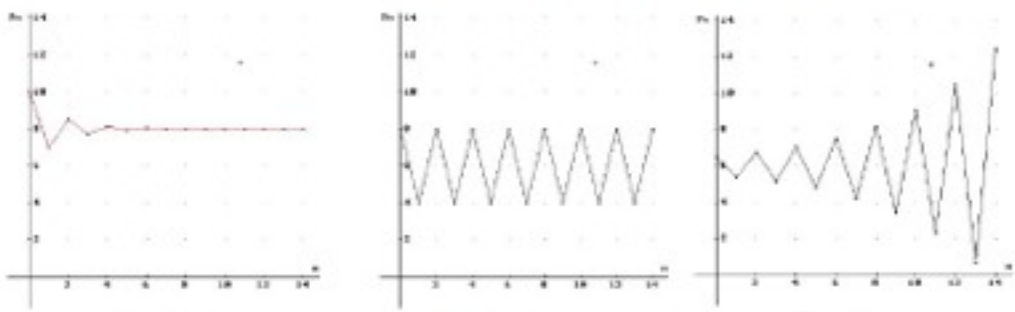


Figure 3.1 Figure 3.2 Figure 3.3

Discussion: In Case 1, $a = -m_s / m_d = -0.5$, $b = (c_d - c_s)/m_d = 12$, and equation (3.4) becomes, $P_{n+1} = -0.5P_n + 12$. The equilibrium price $P_e = b/(1-a) = 8$. The graph of the ordered pairs $(n, P_n), n = 0, 1, \ldots, 14$ is shown in Figure 3.1. The price oscillates around 8, that is the solution is alternately above and below the line $P = 8$. The amplitude of the oscillation decreases and eventually the price P_n approaches 8 as n becomes large. Consequently the equilibrium price is stable.

In Case 2, $a = -1, b = 12$, and $P_e = 6$. The graph of $(n, P_n), n = 0, 1, \ldots, 14$ is shown in Figure 3.2, where the price oscillates between 8 and 4 that is between P_0 and $-P_0 + b$.

In Case 3, $a = -1.2, b = 13.2$, and $P_e = 6$. The graph of $(n, P_n), n = 0, 1, \ldots, 14$ is shown in Figure 3.3, where the price oscillates around $P = 6$. The amplitude of the oscillation increases without bound. The price diverges from the equilibrium price. Consequently, the market price is unstable. Note that this model will fail if the price becomes negative.

(Source: Shahin, 2010)